# Infinity Computing in Global and Local Optimization

Dmitri E. Kvasov and Marat S. Mukhametzhanov

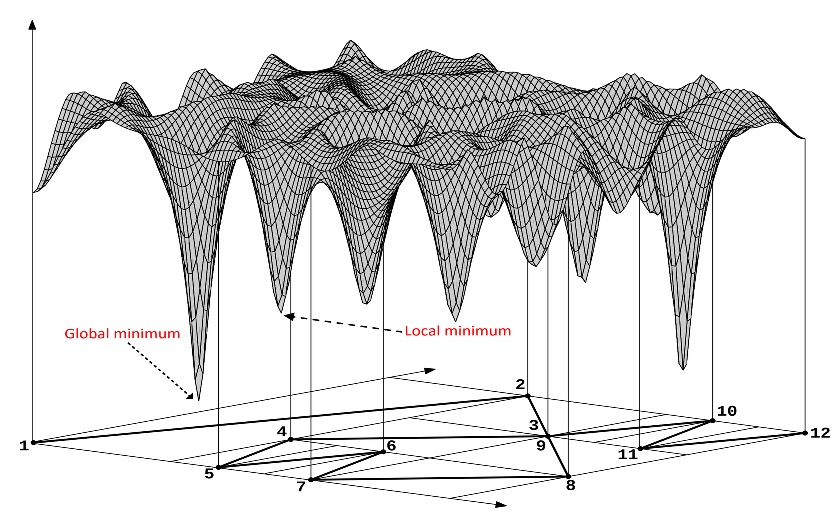
DIMES, University of Calabria, Italy, and Software Department, Lobachevsky Nizhni Novgorod State University, Russia
email: {*kvadim, m.mukhametzhanov*}*@dimes.unical.it*

## Targets of the research

The research is dedicated to application of the Infinity Computer – a new type of a supercomputer able to work **numerically** with infinities and infinitesimals – in global and local optimization with costly and noisy objective functions. Important industrial applications: solution to expensive and ill-conditioned optimization problems in image processing and noisy data fitting.

## Global and Local Optimization

### Expensive global optimization problems



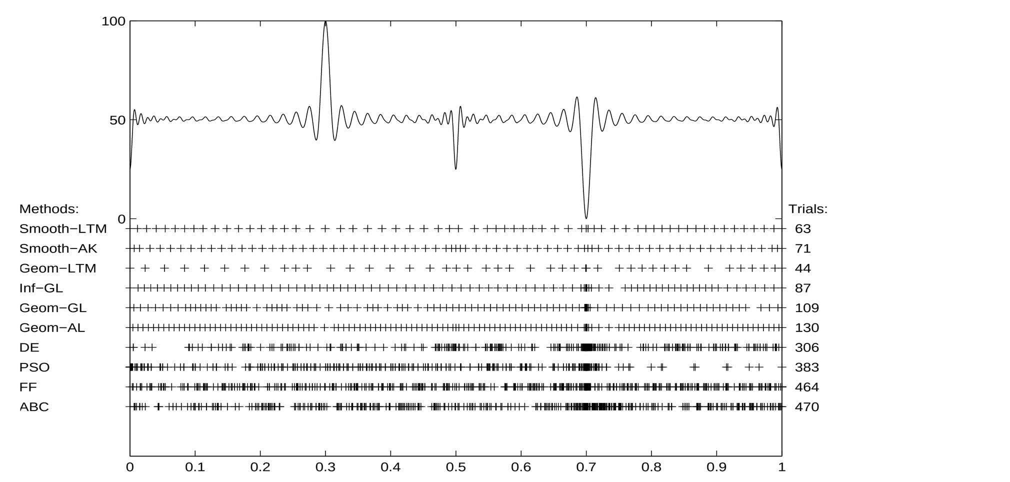

**A challenging problem:** given a limited computational budget, it is required to find a good approximation of the global solution to a multiparametric and multimodal costly objective function subject to nonlinear constraints.
**A promising approach:** extension of univariate methods to the multivariable case by means of diagonal space-filling curves ([9]).

### Applications in noisy data fitting

A general nonlinear regression problem

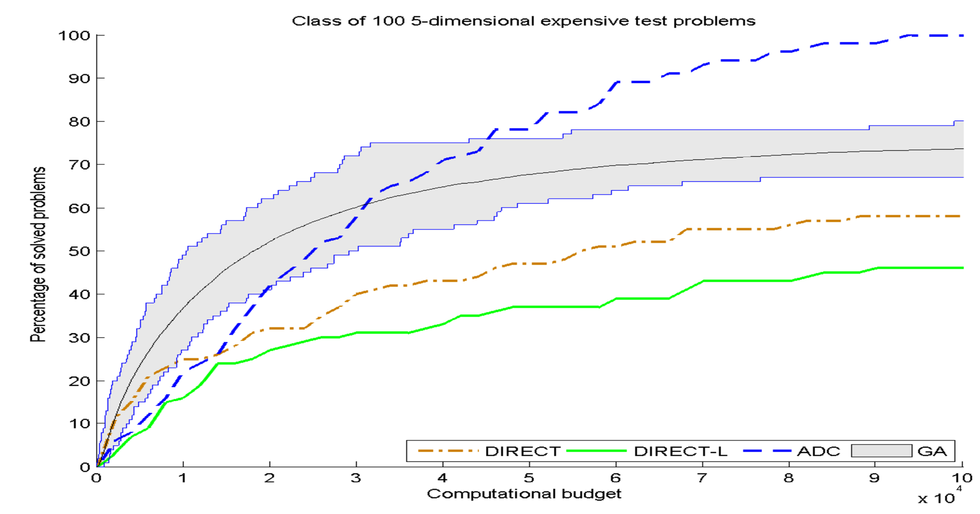
$$f(x) = \sum_{t=1}^{T} (y_t - \eta(x,t))^2 \to \min_{x \in \Omega}, \ \Omega \subset \mathbb{R}^N, \ N = 4q,$$

where $y_t$, $1 \le t \le T$, are real-valued observations corrupted by noise, $\eta(x,t) = \sum_{i=1}^{q} a_i exp(d_i t) sin(2\pi\omega_i t + \phi_i)$, $1 \le t \le T$, [$4q$ parameters to identify, see [3]]




### Metaheuristic vs Deterministic methods

Metaheuristics (as genetic or other nature-inspired algorithms) are often used to study expensive black-box optimization problems.




However, the proposed deterministic methods (e.g., based on adaptive diagonal curves, ADC) demonstrate a much better performance with respect to widely used deterministic (e.g., DIRECT) and metaheuristic (e.g., genetic algorithm, GA) methods (see [4]).

## Infinity Computing

### An astonishing analogy

Numeral system of the amazonian Pirahã tribe.
They can count only **1**, **2**, ***many***:

$$many + 1 = many, \ many + 2 = many,$$
$$many + many = many.$$

Traditional views on infinity:

$$\infty + 1 = \infty, \ \infty + 2 = \infty,$$
$$\infty + \infty = \infty$$

### Grossone

Grossone (①) is the number of elements of the set of natural numbers. The principles of work with ① are the same as with finite numbers (see Ya. Sergeyev. *Arithmetic of Infinity*, CS, 2nd ed 2013):

$$0 \cdot ① = ① \cdot 0 = 0, \ ① - ① = 0, \ \frac{①}{①} = 1, \ ①^0 = 1, \ 1^① = 1, \ 0^① = 0.$$

The non-contradictory nature of the methodology has been proven in [5]. Numeral system allowing one to execute operations with finite, infinite and infinitesimal numbers in a unique framework has been implemented on the Infinity Computer (see the patents [8]).

### Applications

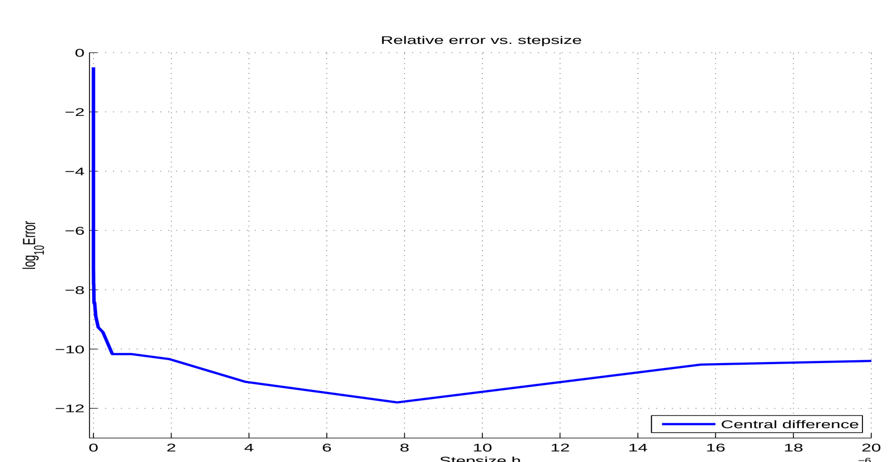
- Global and local optimization
- Numerical differentiation
- Ordinary differential equations
- Turing machines
- Cellular automata
- Set theory
- Mathematical analysis
- Hyperbolic geometry and tiling
- Fractals and percolation, etc. (for details, see references in [7]).

### Numerical differentiation

Suppose that we have a computer procedure $f(x)$ implementing the function $g(x) = \frac{x+1}{x-1}$ and we want to evaluate the value $f'(y)$ at the point $y = 3$. Numerical approximations are used for this purpose on traditional computers:

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}, \ f'(x) \approx \frac{f(x) - f(x-h)}{h},$$
$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}.$$

### Traditional computers – errors




### Infinity Computer – no error

The Infinity Computer executes **numerically** the operations

$$f(3 + ①^{-1}) = (3①^0 + ①^{-1} + 1①^0)/(3①^0 + ①^{-1} - 1①^0) =$$
$$= 2①^0 - 0.5①^{-1} + 0.25①^{-2} - 0.125①^{-3} + 0.0625①^{-4} - \ldots$$

From this numeral, we obtain (see [11])

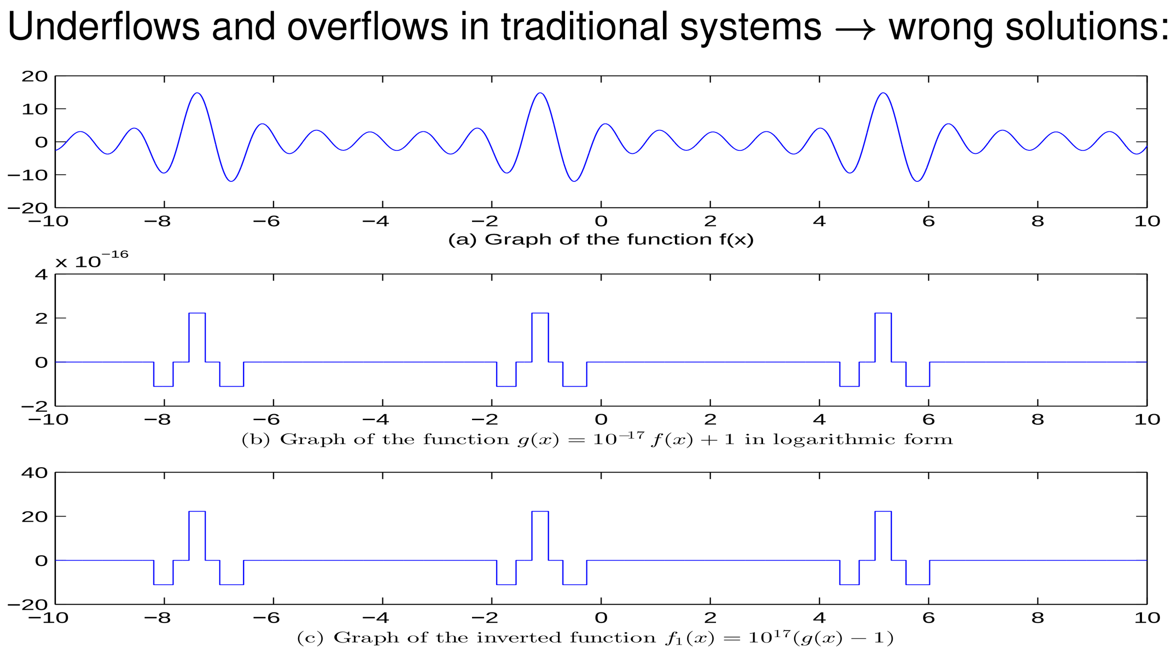
$$f(3) = 2, \ f'(3) = -0.5, \ f''(3) = 2! \cdot 0.25 = 0.5,$$
$$f^{(3)}(3) = 3! \cdot (-0.125) = 0.75,$$

being *exact* values of $f(x)$ and the derivatives at the point $y = 3$.

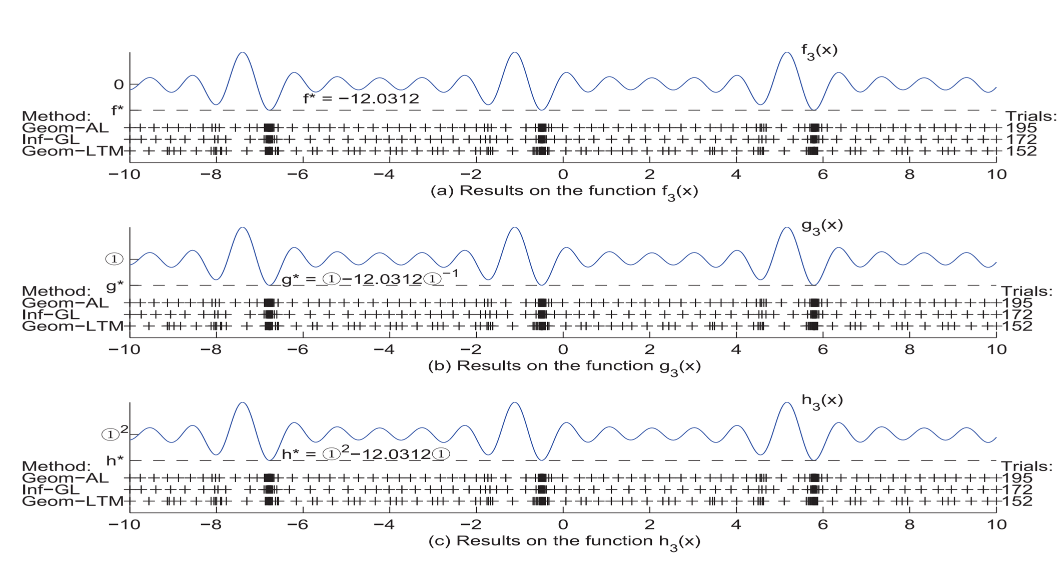
## Infinity Computing in Optimization

### Traditional computers: ill-conditioning

Underflows and overflows in traditional systems → wrong solutions:




### Infinity Computer: well-conditioning




$$f_3(x) = \sum_{k=1}^{5} -k \cdot sin[(k+1)x + k], \ g_3(x) = ①^{-1} f_3(x) + ①, \ h_3(x) = ① f_3(x) + ①^2$$

### Constrained optimization: exact penalty

$$\min_x \frac{1}{2}x_1^2 + \frac{1}{6}x_2^2$$
subject to $x_1 + x_2 = 1$

Penalty approach:

$$\min_x \frac{1}{2}x_1^2 + \frac{1}{6}x_2^2 + \frac{P}{2}(1 - x_1 - x_2)^2.$$

Traditional computers – iterative procedures with different $P$ can return approximated solutions only.
Infinity Computer – exact penalty $P = ①$ (see [1]):

$$x_1^* = \frac{1}{4} - ①^{-1}(\frac{1}{16} - \frac{1}{64}①^{-1} + \ldots), \ x_2^* = \frac{3}{4} - ①^{-1}(\frac{3}{16} - \frac{3}{64}①^{-1} + \ldots)$$

The finite parts of $x_1^*$ and $x_2^*$ give us the exact solution to the original constrained problem: $\overline{x} = (\frac{1}{4}, \frac{3}{4})$

## Obtained results

Infinity Computing has been successfully applied for solving important instances of ill-conditioned optimization problems [2,3]. New powerful multivariable optimization schemes have been proposed [4,6,9,10]: global optimization algorithms based on adaptive diagonal curves, acceleration techniques in derivative-free and smooth global optimization, grossone-based penalty functions in constrained optimization.

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